Recovering aggregate risk aversion from observable variables

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Abstract

This paper introduces a new method to approximate the welfare cost of fluctuations and aggregate risk aversion from observable variables in an economy with agents endowed with heterogeneous risk preferences. Numerical examples show that our estimate of welfare costs is reasonably precise. We apply our methodology to compute the welfare cost of output fluctuations from annual returns of risky and risk-free assets and show that the cost is equivalent to about $2-3\%$ of aggregate consumption.

Keywords: `Business cycle, Heterogeneity, Risk aversion, Risk premium, Utility recovery, Welfare cost.`

Journal of Economic Literature Classification Numbers: `G12, G13, E32.`

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1 Introduction

Utility recovery theory seeks to find pricing kernel and risk preferences from observable variables such as prices and aggregate level of output/consumption. Finding a pricing kernel and utility functions (i.e., risk preferences) are identical under the assumption of a representative agent. Therefore, it is common practice to assume the existence of a representative agent in the theoretical literature on utility recovery (see Ross (2015) and Kubler and Polemarchakis (2017)). However, under heterogeneous risk preferences, it is impossible to find utility functions at the level of individuals without information on consumption/endowment distributions, which are not easily observable. Thus, it is worth investigating utility recovery theory under conditions of a heterogeneous risk aversion considering that (1) utility functions reveal more information than a pricing kernel,[1] and (2) empirical evidence supports heterogeneous risk aversion.

Since a representative agent, in general, does not exist under heterogeneous risk aversion, we need to approach recovery theory using an alternative method. In this paper, rather than directly deriving utility functions, we use first-order approximation to derive the aggregate welfare cost of risk. We also report an approximate measure of risk aversion based on the estimated welfare cost of risk.

The first-order approximation proposed in this paper can be applied to calculate the welfare cost of business cycles shown in Lucas (1987), who measured the welfare cost of business cycles in terms of consumption goods. This approach has become an important research tool and has received attention in several recent research papers (see Tallarini (2000) and Schulhofer-Wohl (2008) among the recent papers). A common feature in the literature is the reliance on the assumption that a representative agent has CRRA preferences in estimating the welfare costs. However, there are two main obstacles in measuring the cost accurately following this approach: (a) wide disagreement among economists over the correct measure of the representative agent's cardinal utility index (i.e., relative risk aversion); and (b) given

[1] Specifically, a pricing kernel provides information on asset prices but not on the welfare cost of risk or consumption demand functions that can be derived from utility functions.

the heterogeneity in agents' preferences, the aggregate utility function may not exist in general, so the aggregate welfare cost cannot be computed using the conventional representative agent approach.

Empirical estimation of risk aversion has been a difficult task and has led to considerable controversy around its precise and reliable measurement. In view of the wide disparity in the estimates of risk aversion, the welfare cost of a business cycle varies based on the selection of the risk-aversion parameter value. For example, Lucas (2003) estimates the welfare cost of a business cycle are 0.05% of consumption based on the relative-risk-aversion parameter being 1, while Tallarini (2000) estimates the welfare costs to be up to 10% of consumption based on the parameter being more than 50.[2]

Considering the lack of consensus on the precise estimate of the risk aversion parameter, it is natural that there is no agreement on the measure of the welfare cost. In this paper, we propose a method to measure the welfare cost in situations where information about the aggregate risk aversion is not available. Instead of relying on the cardinal utility function and aggregate output data, we estimate the welfare cost from the data of prices and output. We first show that information about risk aversion is incorporated in the prices in such a way that with an increase in the risk aversion, the price and output are more negatively correlated. Therefore, the welfare cost increases with increased risk aversion.

We also show that our proposed first-order approximation can be expressed in terms of the prices of risky and risk-free assets. Assuming that the annual return of bonds is 2% and the annual average return of stock is 8%, our result shows that the welfare cost is around 2.8%, which is considerably higher than that proposed by Lucas (2003). Many economists have attempted to explain the disparity between the asset prices (e.g. equity risk premium) and Lucas's low level of welfare cost of business cycles. Previous approaches to narrow the gap include assuming different risk preferences (e.g., habit formation) or heterogeneous risk preferences. However, the merit of the methodology introduced in this paper is that we can

[2]Tallarini (2000) uses preferences of Epstein and Zin (1989) with the inter-temporal substitution elasticity of 1 and provides the comparative statics on how the equity premium and risk-free rate varies with relative-risk-aversion parameter. The welfare cost with the relative-risk-aversion parameter that provides an empirically-matched value of equity risk premium is 10% of the average consumption.

avoid any controversial issue related to heterogeneity and utility functions in deriving the welfare cost.

Our result could be applied to an economy with heterogeneous agents. Vast empirical and experimental evidence indicates that individuals' risk preferences are not identical across gender, age, and regions.[3] An economy with such heterogeneous agents cannot be modeled using a conventional representative agent approach. However, we show that the sum of all individuals' welfare costs due to uncertainty can be derived even in the absence of the information of agents' utility functions. In an economy with heterogeneous agents, the welfare cost of each individual has a distinct general equilibrium effect that arises from price changes due to increased uncertainty. With a sufficiently small risk, the individual's general equilibrium effect simply becomes the individual's excess demand multiplied by the changed prices. Therefore, in the presence of a small risk, the aggregate general equilibrium effects vanishes at the economy level.

The small risks approach originates from the classic research by Pratt (1964), in which the relationship between risk-aversion and risk-premium is revealed in a single-agent economy. In addition, empirical data shows that the business cycle fluctuations have a standard deviation of less than 5% around the consumption path, which further justifies our results. We test our results based on the following two examples of the welfare cost of a business cycle in an economy consisting of (a) a representative agent (as in Lucas (2003)), and (b) agents having heterogeneous risk preferences (as in Schulhofer-Wohl (2008)). Both tests indicate that the measurement errors from the proposed first-order approximation are significantly small.

Our paper complements the recovery of cardinal utility or pricing kernel that has been explored in several research papers in economics and finance.[4] Our model admits hetero-

[3] See Eckel and Grossman (2008), Jianakoplos and Bernasek (1998), and Harbaugh et al. (2002) for evidence. The heterogeneous risk aversion model is also used in asset pricing literature. See Dumas (1989), Wang (1996), Chan and Kogan (2002), and Bhamra and Uppal (2013). The literature shows that the model with heterogeneous agents results in different equilibrium outcomes, but it is empirically more consistent, than the representative agent model.

[4] See Mas-Colell (1977), Cuoco and Zapatero (2000), Kubler et al. (2014), Ross (2015), Borovička et al. (2016), Kubler and Polemarchakis (2017), Walden (2017), Jensen et al. (2019), for classic as well as recent references.

geneous risk preferences. With heterogeneous risk aversion, it is not possible to derive the single welfare cost of fluctuations due to the heterogeneous general equilibrium effects.[5] We show that with small fluctuations, the first-order approximation methodology enables us to aggregate the general equilibrium effects with different risk preferences. Related to the literature on business cycles, our paper complements the work by Alvarez and Jermann (2004), in which the cost of a business cycle is derived by relying on asset prices and consumption risk premia. They measured the marginal cost of consumption fluctuations from the asset prices and calculated the welfare cost with the first-order approximation.[6] However, their results are different from ours in two ways. First, their results are based on the assumption that there exists a representative agent. Second, they defined the marginal cost of consumption fluctuations and derived the corresponding first-order approximations according to their defined cost. Consequently, their first-order approximation is not directly derived from the equilibrium allocations. In contrast, we derive the first-order approximation from general-equilibrium analysis and investigate how the asset prices are related to the approximation.

The rest of the paper is organized as follows. In Section 2, we introduce the main result of this paper showing how the output-fluctuation cost and risk aversion can be approximately derived from market prices and aggregate uncertainty. Section 3 presents an example indicating (a) how the result can be applied in calculating the business cycle cost and (b) how the of approximation introduced in this paper is quite accurate. Section 4 presents another example showing the aggregate risk aversion can be derived from the heterogeneous economy. Section 5 applies our approximation to compute the welfare cost of output fluctuations using prices of risky and risk-free assets. Finally, concluding remarks are presented in Section 6.

[5] Specifically, Ross (2015) assumes that there is a single representative investor and Kubler and Polemarchakis (2017) allows heterogeneous beliefs while maintaining an identical cardinal utility index.

[6] Alvarez and Jermann (2004) calculates the welfare cost based on the lifetime utility rather than single-period utility. Because such assets providing the lifetime stable consumption steams do not exist, Alvarez and Jermann (2004) computes them indirectly from existing asset prices

2 Main result

In this section, we assume that there are N agents in the economy where $H = \{1, 2, \cdots, H\}$. Each agent faces his/her own risk. Agent h's endowment where $h \in H$ is defined as

$$\widetilde{\omega}_h = \omega_h + \sqrt{\sigma}\widetilde{z}_h. \tag{1}$$

As in Pratt (1964), $\widetilde{z}_h$ is a random variable whose expected value is zero. If the mean-preserving parameter (σ) is 1, our notation is similar to Pratt (1964). In contrast to Pratt (1964) in which the variance of $\widetilde{z}_h$ approaches zero, a "small"risk premium, in this paper, is defined based on the assumption that σ approaches zero.[7]

In a complete market, agents have risk-sharing opportunities so the equilibrium consumption is different from endowments. We denote $\widetilde{x}_h$ as the equilibrium consumption of agent h. Agent h's vNM utility $u_h(\cdot)$ is continuously differentiable, strictly increasing and strictly concave. Given a random variable price $\widetilde{p}$, each agent maximizes the expected utility:

$$\left.\begin{array}{rl} \max\limits_{\widetilde{x}_h} & \mathbb{E}\, u_h(\widetilde{x}_h) \\ \text{subject to} & \mathbb{E}\,\widetilde{p}\widetilde{x}_h = \mathbb{E}\,\widetilde{p}\widetilde{\omega}_h. \end{array}\right\}. \tag{2}$$

The market clearing condition is

$$\sum_{h \in H} \widetilde{x}_h = \sum_{h \in H} \widetilde{\omega}_h.$$

Agent h's risk premium (π_h) in general equilibrium is defined as

$$u_h(\omega_h - \pi_h) = \mathbb{E} u_h(\widetilde{x}_h). \tag{3}$$

[7] An intuitive explanation for introducing parameter σ is to extend Pratt's one-agent model to a many-agents economy. With many consumers, we have many $\widetilde{z}$'s as $(\widetilde{z}_1, \widetilde{z}_2, \cdots, \widetilde{z}_H)$ as a H-dimensional vector. The parameter (σ) helps us control all of $\widetilde{z}$'s. We define the uncertainty as $\sqrt{\sigma}(\widetilde{z}_1, \widetilde{z}_2, \cdots, \widetilde{z}_H)$ instead of $(\widetilde{z}_1, \widetilde{z}_2, \cdots, \widetilde{z}_H)$. Then, as $\sqrt{\sigma} \to 0$, all uncertainties go to zero. Needless to say, if there is one consumer, we do not need parameter σ.

Differentiating *Eq.* (3) with respect to σ, we have

$$-u_h'(\omega_h - \pi_h)\frac{d\pi_h}{d\sigma} = \mathbb{E}u_h'(\widetilde{x}_h)\frac{d\widetilde{x}_h}{d\sigma}. \tag{4}$$

In equilibrium, the normalized price $\widetilde{p}$ is

$$\widetilde{p} = \frac{u_h'(\widetilde{x}_h)}{\mathbb{E}u_h'(\widetilde{x}_h)} \text{ for all } h \in H. \tag{5}$$

From *Eqs.* (4) and (5), we have

$$-\frac{u_h'(\omega_h - \pi_h)}{\mathbb{E}u_h'(\widetilde{x}_h)}\frac{d\pi_h}{d\sigma} = \mathbb{E}\widetilde{p}\cdot\frac{d\widetilde{x}_h}{d\sigma}. \tag{6}$$

The budget constraint for agent h is

$$\mathbb{E}\widetilde{p}\widetilde{x}_h = \mathbb{E}\widetilde{p}\widetilde{\omega}_h. \tag{7}$$

Implicitly differentiating the budget constraint with σ, we have

$$\begin{aligned}\mathbb{E}\frac{d\widetilde{p}}{d\sigma}\widetilde{x}_h + \mathbb{E}\widetilde{p}\frac{d\widetilde{x}_h}{d\sigma} &= \mathbb{E}\frac{d\widetilde{p}}{d\sigma}\widetilde{\omega}_h + \mathbb{E}\widetilde{p}\frac{d\widetilde{\omega}_h}{d\sigma} \\ &= \mathbb{E}\frac{d\widetilde{p}}{d\sigma}\widetilde{\omega}_h + \mathbb{E}\widetilde{p}\widetilde{z}_h\frac{d(\sqrt{\sigma})}{d\sigma}.\end{aligned} \tag{8}$$

From Eqs. (6) and (8), we have

$$\frac{d\pi_h}{d\sigma}\frac{u_h'(\omega_h - \pi_h)}{\mathbb{E}u_h'(\widetilde{x}_h)} = \mathbb{E}(\widetilde{x}_h - \widetilde{\omega}_h)\cdot\frac{d\widetilde{p}}{d\sigma} - \frac{\mathbb{E}\widetilde{p}\widetilde{z}_h}{2\sqrt{\sigma}}. \tag{9}$$

Limiting in both sides of *Eq.* (6), we have

$$\lim_{\sigma\to 0}\frac{d\pi_h}{d\sigma}\frac{u_h'(\omega_h - \pi_h)}{\mathbb{E}u_h'(\widetilde{x}_h)} = \lim_{\sigma\to 0}\mathbb{E}(\widetilde{x}_h - \widetilde{\omega}_h)\cdot\frac{d\widetilde{p}}{d\sigma} - \lim_{\sigma\to 0}\frac{\mathbb{E}\widetilde{p}\widetilde{z}_h}{2\sqrt{\sigma}}. \tag{10}$$

We know that $\lim_{\sigma\to 0} \frac{u_h'(\omega_h-\pi_h)}{\mathbb{E}u_h'(\widetilde{x}_h)}$ is finite and converges to 1 because

$$\lim_{\sigma\to 0}(\omega_h-\pi_h)=\omega_h=\lim_{\sigma\to 0}(\widetilde{x}_h).$$

Therefore, *Eq.* (10) can be written as

$$\lim_{\sigma\to 0}\frac{d\pi_h}{d\sigma}=\underbrace{\lim_{\sigma\to 0}\mathbb{E}(\widetilde{x}_h-\widetilde{\omega}_h)\cdot\frac{d\widetilde{p}}{d\sigma}}_{\text{General equilibrium effect}}+\underbrace{\lim_{\sigma\to 0}\left(-\frac{\mathbb{E}\widetilde{p}\widetilde{z}_h}{2\sqrt{\sigma}}\right)}_{\text{Partial equilibrium effect}}. \tag{11}$$

We denote the first term $\mathbb{E}(\widetilde{x}_h-\widetilde{\omega}_h)\frac{d\widetilde{p}}{d\sigma}$ as the general equilibrium effect, while the second term $-\frac{\mathbb{E}\widetilde{p}\widetilde{z}_h}{2\sqrt{\sigma}}$ is the partial equilibrium effect. The general equilibrium effect considers the price change due to increased uncertainty, $\frac{d\widetilde{p}}{d\sigma}$. In contrast, in a partial equilibrium effect, the price $\widetilde{p}$ is assumed to be unaffected by an increase in risk, i.e., $\frac{d\widetilde{p}}{d\sigma}=0$. In the following proposition, we show that the general equilibrium effects in the aggregate level with a small risk vanishes, so the total welfare cost can be computed only by the partial equilibrium effects:

Proposition 1. *The first-order term of the aggregate cost (i.e., the sum of all individuals' risk premia) is expressed as*

$$\lim_{\sigma\to 0}\sum_{h\in H}\frac{d\pi_h}{d\sigma}=-\lim_{\sigma\to 0}\frac{\mathbb{E}\widetilde{p}\widetilde{z}}{2\sqrt{\sigma}} \tag{12}$$

and is strictly positive for any non-degenerate random variable $\widetilde{z}\left(=\sum_{h\in H}\widetilde{z}_h\right)$.

Proof. From equation (11), we have

$$\begin{aligned}\sum_{h\in H}\lim_{\sigma\to 0}\frac{d\pi_h}{d\sigma} &= \sum_{h\in H}\lim_{\sigma\to 0}\mathbb{E}\left(\widetilde{x}_h-\widetilde{\omega}_h\right)\frac{d\widetilde{p}}{d\sigma}-\sum_{h\in H}\lim_{\sigma\to 0}\frac{\mathbb{E}\widetilde{p}\widetilde{z}_h}{2\sqrt{\sigma}}\\ &= \lim_{\sigma\to 0}\sum_{h\in H}\mathbb{E}\left(\widetilde{x}_h-\widetilde{\omega}_h\right)\frac{d\widetilde{p}}{d\sigma}-\lim_{\sigma\to 0}\left(\sum_{h\in H}\frac{\mathbb{E}\widetilde{p}\widetilde{z}_h}{2\sqrt{\sigma}}\right)\\ &= \lim_{\sigma\to 0}\mathbb{E}\sum_{h\in H}\left(\widetilde{x}_h-\widetilde{\omega}_h\right)\frac{d\widetilde{p}}{d\sigma}-\lim_{\sigma\to 0}\mathbb{E}\left(\frac{\sum_{h\in H}\widetilde{p}\cdot\widetilde{z}_h}{2\sqrt{\sigma}}\right)\\ &= \lim_{\sigma\to 0}\mathbb{E}\sum_{h\in H}\left(\widetilde{x}_h-\widetilde{\omega}_h\right)\frac{d\widetilde{p}}{d\sigma}-\lim_{\sigma\to 0}\mathbb{E}\left(\frac{\widetilde{p}\sum_{h\in H}\widetilde{z}_h}{2\sqrt{\sigma}}\right).\end{aligned} \tag{13}$$

Since $\sum_{h\in H}\left(\widetilde{x}_h-\widetilde{\omega}_h\right)=0$ by a market clearing condition, the first term in equation (13) vanishes, i.e.,

$$\lim_{\sigma\to 0}\mathbb{E}\sum\nolimits_{h\in H}\left(\widetilde{x}_h-\widetilde{\omega}_h\right)\frac{d\widetilde{p}}{d\sigma}=0.$$

□

With a small risk, the aggregate general equilibrium effect vanishes to zero. Individuals can have a positive (negative) general equilibrium effect if their excess demand $(\widetilde{x}_h-\widetilde{\omega}_h)$ is positively (negatively) correlated with price changes $(d\widetilde{p}/d\sigma)$. However, because the sum of excess demand is zero by the market clearing conditions, the aggregate general equilibrium effect is zero. Distinct from the aggregate general equilibrium effect, the aggregate partial equilibrium effects need not vanish as expressed in terms of the expected value of the product of aggregate uncertainty and the market prices.

Consequently, Proposition 1 indicates that for a small risk, the aggregate cost can be represented by the market price $(\widetilde{p})$ and the aggregate uncertainty $(\widetilde{z})$, but it does not depend on individual uncertainty $(\widetilde{z}_h)$. Thus, we can get the aggregate welfare cost with first-order approximation from the price and aggregate uncertainty. Specifically, assuming that the variance of $\widetilde{z}$ is small (instead of assuming that σ is small) and $\sigma=1$, from *Eq.* (12) of Proposition 1 we can derive the first-order approximate aggregate cost of risk $(\pi=\sum_{h\in H}\pi_h)$:

Corollary 1. *When the observed price is* $\widetilde{p}$ *and the observed output volatility is* $\widetilde{z}(=\sqrt{\sigma}\widetilde{z})$ *where*

$E[\widetilde{z}] = 0$, *the first-order approximation of the aggregate cost is*

$$\pi \approx -\frac{\mathbb{E}\widetilde{p}\widetilde{z}}{2}. \tag{14}$$

Proof. From Proposition 1, the first-order approximate cost is

$$\pi \approx \sum_{h \in H} \frac{d\pi_h}{d\sigma} \times \sigma,$$

When $\sigma = 1$ (because we have $\widetilde{z} = \sqrt{\sigma}\widetilde{z}$), the first-order approximation is $-\mathbb{E}\widetilde{p}\widetilde{z}/2$. $\square$

Examples 1 and 2 in the next sections show that the first-order approximation in *Eq.*(14) in Corollary 1 is fairly accurate based on the literature on the cost of a business cycle. From *Eq.*(12) in Proposition 1, we also know that the aggregate risk premium in this paper can be defined without information about agents' preferences. This is because information about heterogeneous agents' risk aversion (or utility) is incorporated into the prices: as the agents become more risk averse, the price is more negatively correlated with the endowment, which results in a high risk premium in *Eq.*(12).

The result in Corollary 1 also implies that once we specify a utility function, we can approximate the risk parameters of the utility function (e.g., absolute or relative risk aversion parameters) from the data. Using the same logic in constructing Eq.(14), we calculate the first-order approximation of relative risk aversion if we assume that the utility function is in the class of CRRA as shown in the following corollary:

Corollary 2. *When the observed price is $\widetilde{p}$ and the observed output volatility is $\widetilde{z}$, the first-order approximation of relative risk aversion (γ) is*

$$\gamma \approx -\frac{\omega\left(\mathbb{E}\widetilde{p}\widetilde{z}\right)}{var(\widetilde{z})}, \tag{15}$$

where $\omega = \sum_{h \in H} \omega_h$.

Proof. Assuming that there exists a representative agent with constant relative risk aversion (γ), the aggregate risk premium from Pratt (1964) is

$$\pi \approx -\gamma \frac{var(\sqrt{\sigma}\tilde{z}/\omega)}{2}\omega = -\gamma\sigma\frac{var(\tilde{z})}{2\omega},$$

which implies that

$$\frac{d\pi}{d\sigma} \approx \gamma \frac{var(\tilde{z})}{2\omega} \tag{16}$$

where γ represents the relative risk aversion. Equating *Eqs.*(12) and (16), we can obtain

$$\gamma \approx \left[-\frac{\omega\left(\mathbb{E}\tilde{p}\tilde{z}\right)}{var(\tilde{z})\sqrt{\sigma}}\right], \tag{17}$$

With $\sigma = 1$, we have $\gamma \approx -\,\omega\left(\mathbb{E}\tilde{p}\tilde{z}\right)/var(\tilde{z})$ from *Eq.*(17). □

Corollary 2 shows how to approximate the relative risk aversion under the assumption that a representative agent with CRRA preferences exists. Example 2 in Section 4 shows how aggregate risk aversion can be derived under the heterogeneity of risk aversion. However, our result places no restriction on the functional forms of utility. Assuming different utility functions with habit formation or absolute risk aversion, we can also estimate the parameters. For example, with a representative agent with absolute risk aversion, we can obtain a first-order approximation of absolute risk aversion (a):

$$a \approx -\frac{\mathbb{E}\tilde{p}\tilde{z}}{var(\tilde{z})}, \tag{18}$$

which can be shown in the same way in Corollary 2.

3 The cost of a business cycle in Lucas (2003)

In order to compare the first-order-approximation result with conventional business-cycle literature, in this example we use a homogeneous CES utility function even though our result can be applied to any heterogeneous-agent economy. This example shows that the proposed approximation of the welfare cost is highly precise.

Example 1. We assume that the aggregate endowment is defined as a binary distribution with two states α and β with equal probability such as

$$\widetilde{\omega} = \omega + \sqrt{\sigma} \cdot \widetilde{z},$$

where the aggregate endowment in state α is $-\omega \times s$ $(z(\alpha) = -\omega \times s)$ and that in state β is $\omega \times s$ $(z(\beta) = \omega \times s)$. As mentioned in the previous section, we assume $\sigma = 1$ and compare the exact value of the risk cost and approximate cost with $\sigma = 1$. Then, s represents the relative standard deviation of the aggregate endowment. In an economy with identical CES utilities whose relative risk aversion is γ, the distribution of the endowment does not affect the equilibrium price. Therefore, the equilibrium price is

$$\widetilde{p} = \{p(\alpha), p(\beta)\} = \left\{ \frac{2(1+s)^{\gamma}}{(1+s)^{\gamma} + (1-s)^{\gamma}}, \frac{2(1-s)^{\gamma}}{(1+s)^{\gamma} + (1-s)^{\gamma}}; 50\%, 50\% \right\}. \tag{19}$$

The observable variables are $\widetilde{z}$ and $\widetilde{p}$. With this setting, from *Eq.*(14) in Corollary 1, we have the first-order approximation of the welfare cost given by

$$\pi \approx -\frac{\mathbb{E}\widetilde{p}\widetilde{z}}{2} = \left(\frac{s}{2}\right) \left(\frac{(1+s)^{\gamma} - (1-s)^{\gamma}}{(1+s)^{\gamma} + (1-s)^{\gamma}}\right) \omega. \tag{20}$$

For example, using the same parameter in Lucas (2003) for the GDP fluctuation from (1947-2001), the standard deviation of real consumption is around 3.2% (i.e., $s = 0.032$). Assuming the utility is log linear (i.e., $\gamma = 1$) following Lucas (2003), the approximate welfare cost in our approach from *Eq.*(20) is $\pi = \omega \times 5.12 \times 10^{-4} = \omega \times 0.0512\%$. Assuming that a representative agent exists, the aggregate cost $\pi (= \sum_{h \in H} \pi_h)$ can be computed as:

$$\log(\omega - \pi) = \frac{1}{2} \log(\omega + 0.032\omega) + \frac{1}{2} \log(\omega - 0.032\omega),$$

which gives $\pi = \omega \times 0.0512131\%$, and is very close to the approximation with error being just

$0.026\% = (0.0512131 - 0.0512)/0.0512131$.[8]

Table 1: Comparison of our approximation and Lucas (2003)

	$\gamma = 1$		$\gamma = 5$		$\gamma = 10$	
s	Exact	Approx.	Exact	Approx.	Exact	Approx.
3.2%	0.051%	0.051%	0.255%	0.254%	0.504%	0.495%
5%	0.133%	0.125%	0.621%	0.613%	1.201%	1.156%
10%	0.509%	0.5%	2.434%	2.317%	4.424%	3.815%
15%	0.126%	0.125%	5.307%	4.789%	8.843%	6.804%

The welfare cost (in percentage) under homogeneous risk preferences. In each box, the first number represents the exact value of the welfare cost and the second number represents the first-order approximation.

Table 1 indicates how the error (in percentage terms) changes as standard deviation (s) varies from 3.2% to 15% and the relative risk aversion takes values of 1, 5, and 10. The error is close to zero when the uncertainty is small as shown in Proposition 1. Even when the standard deviation is high, the error is reasonably small: when the standard deviation is high at 10%, the errors are about 0.25% (if $\gamma = 1$), 4.8% (if $\gamma = 5$), 13.8% (if $\gamma = 10$), respectively. In the literature, the standard deviations of a business cycle for both output and consumption are reported to be less than 5%. Given this, we can conclude that the error in our approximation is negligible.

4 Heterogeneous risk preferences and aggregate cost/risk aversion

In this section, we apply our first-order approximation to an example discussed in Schulhofer-Wohl (2008) to investigate the cost of a business cycle under heterogeneous risk aversion. We show that even with heterogeneous preferences, our estimated cost is significantly accurate.

Example 2. Assume that there are two agents with relative risk aversions of 4 and 1, respectively. There are two discrete states, α and β with equal probabilities. Assuming that the

[8] The error is calculated as

$$\text{Error} = \frac{\text{Exact value - Approximation}}{\text{Exact value}}.$$

standard deviation of the annual output fluctuation is 2%, the aggregate output levels in the two states are 1.02 and 0.98, respectively. The first agent is endowed with $w_1 \in (0.02, 0.98)$ in both states, and the second agent is endowed with $1.02 - w_1$ in state α and $0.98 - w_1$ in state β. We have numerically solved the competitive equilibrium prices and allocations and determined each individual's welfare cost, π_1 and π_2. The approximate welfare cost (π) and aggregate risk aversion (γ) are calculated from *Eq.*(14), i.e., $-\mathbb{E}\widetilde{p}\widetilde{z}/2$ and *Eq.*(15), i.e., $-\omega\mathbb{E}\widetilde{p}\widetilde{z}/var(\widetilde{z})$ respectively. Table 2 summarizes the results and shows the individual costs of risk, aggregate cost, approximated cost, and approximated relative risk-aversion for three distinct values of w_1.

Table 2: Comparison of our approximation and Schulhofer-Wohl (2008)

	From competitive equilibrium			From first-order approximation	
w_1	π_1	π_2	$\pi_1 + \pi_2$	π (Welfare cost)	γ (Risk aversion)
0.1	-0.00584469%	0.0222087%	0.0216243%	0.0216217%	1.08109
0.5	-0.0064034%	0.0384143%	0.0320109%	0.0320029%	1.60014
0.9	-0.0427576%	0.104421%	0.0616629%	0.0616094%	3.08047

The welfare cost in percentage under heterogeneous risk preferences

Comparing the two welfare costs – exact welfare cost of $\pi_1 + \pi_2$ and approximate welfare cost of π, we know from Table 2 that the measurement error is ignorably small. We observe that even when we vary the first agent's relative risk aversion between 1 and 10, and the endowment between 0.1 and 0.9, the measurement error is still very small, i.e., less than 0.01%.

5 Using asset prices to estimate the cost of business cycles

From Corollary 1, we know that we can estimate the cost of aggregate uncertainty from prices ($\widetilde{p}$) and fluctuations ($\widetilde{z}$). Although the data for the aggregate fluctuation ($\widetilde{z}$) is commonly available, it is difficult to directly obtain the date for the prices ($\widetilde{p}$). Therefore, this section describes how to derive the welfare cost from an available (observable) asset price date. There are two main assets in the market: a risk-free asset and a risky asset. The return of a risk-free asset is usually derived from government-issued bonds, while that of a risky asset is from the stock market. The following proposition shows that the welfare cost can be computed using the prices of risky and risk-free assets.

Proposition 2. *The first-order approximation of the welfare cost of output fluctuations is*

$$\pi \approx \omega \times \frac{1 - \frac{P_R}{P_F}}{2} \tag{21}$$

where P_R *is the price of a risky asset whose expected payoff is 1, and* P_F *is the price of a risk-free asset whose payoff is 1.*

Proof. From corollary 1, the first-order approximation is

$$\pi \approx -\frac{\mathbb{E}\widetilde{p}\widetilde{z}}{2},$$

which is, in turn, equivalent to

$$\pi \approx -\frac{\mathbb{E}\widetilde{p}\widetilde{\omega} - \mathbb{E}\widetilde{p}\omega}{2} = \frac{\mathbb{E}\widetilde{p}\omega - \mathbb{E}\widetilde{p}\widetilde{\omega}}{2} = \omega\frac{\mathbb{E}\widetilde{p} - \mathbb{E}\widetilde{p}\widetilde{\omega}/\omega}{2} \tag{22}$$

Because $\mathbb{E}\widetilde{p} = 1$, from Eq. (22) we have

$$\pi \approx \omega\frac{1 - (\mathbb{E}\widetilde{p}\widetilde{\omega}/\omega)/\mathbb{E}\widetilde{p}}{2}. \tag{23}$$

In Eq. (23), $\mathbb{E}\widetilde{p}\widetilde{\omega}/\omega$ represents the price of the risky asset whose expected return is 1, while $\mathbb{E}\widetilde{p}$ is the price of risk-free asset whose return is 1. □

From the ratio of the returns on risk-free and risk assets, we can derive the ratio of two prices in Eq. (21). For example, assuming that the expected annual gross return of a risky asset is 108% $(= R_R)$ and the expected annual net return of the risk-free asset is 102% $(= R_F)$, the welfare cost from Proposition 2 is

$$\pi/\omega \approx \frac{1 - R_F/R_R}{2} = \frac{1 - 1.02/1.08}{2} \approx 2.8\% \tag{24}$$

The annual return of risk-free and risky assets are usually derived from short-term government-issued bonds and the stock market (e.g. the S&P 500), respectively. Depending on what data are selected, the rates can be different but most economists would agree that the annual net return on a risk-free asset is in the range of 1~2.5% and that on a risky-asset is in the range of 6~8%. Based on these ranges, the approximated welfare cost is in the range of 1.7~3.2%.

The value of 2.8% as the welfare cost corresponds to the value of relative risk aversion, 54~55, assuming that the standard deviation of the business cycle fluctuations is 3.2%:

$$\gamma \approx -\frac{\omega\left(\mathbb{E}\widetilde{p}\widetilde{z}\right)}{var(\widetilde{z})} = \frac{0.028 \times 2}{(0.032)^2} = 54.7.$$

Calculating the relative risk aversion brings us back to the equity risk premium puzzle. The approach of the puzzle starts from the assumption that a representative agent with CRRA preferences exists. However, we calculate the welfare cost without any information about the risk preferences. The value for relative-risk-aversion is up to 54 only when we apply the welfare cost to the CRRA preference. Under different risk preferences, we would get different parameter values.

The value of 2.8% as the welfare cost is considerably larger than that from the representative-agent model (e.g. Lucas (2003)).There are several possible explanations for this large gap. First, the representative-agent CRRA preferences may not simulate market risk behaviors well. It is possible that the individuals' risk preferences are quite different from CRRA preferences. For example, with habit formation instead of CRRA preferences, high welfare costs would be obtained even with a small risk parameter value. Another explanation is that consumers might have heterogeneous risk preferences. Some consumers with high risk aversion might increase the welfare cost and, which would influence the equity premium.

6 Concluding remarks

This paper introduces an alternative method to calculate the welfare cost of output fluctuations and risk aversion without the information about utility functions. The model in this paper allows heterogeneity of risk preferences. Since the information about consumers' risk preferences is contained in the price-level volatility, the welfare cost can be calculated directly from the price-level volatility. Given that estimates of the aggregate risk aversion vary widely as reported in the literature, our cost calculation could be useful in advancing the research on business cycles, asset pricing, and macroeconomics.

We apply our approximation to compute the welfare cost of output fluctuations using asset prices. When the premium is 6% and the annual return of the risk-free asset is 2%, the welfare cost is equivalent to 2.8% of the average consumption. Based on the CRRA utility function, 2.8% of the welfare cost corresponds to a relative risk aversion of 54, which is considerably lager than the empirically accepted value. This disparity between the equity risk premium and the level of risk aversion is an important research question. However, our approach enables us to compute the welfare cost without specifying the utility function, which allows us to avoid the controversial issue of the equity risk premium puzzle.

This paper is also related to the literature on business cycle costs with heterogeneous risk preferences. In particular, Schulhofer-Wohl (2008) showed that heterogeneity in risk aversion creates more opportunities for trade, thus reducing the welfare cost of business cycles for every agent. Therefore, we know that the assumption of representative agents can be misleading in the estimation of the exact welfare cost in Schulhofer-Wohl (2008). Considering that it is extremely difficult to measure the distribution of heterogeneous risk preferences, our proposed methodology can provide a new method to understand the exact welfare cost under heterogeneity.

This paper considers a small risk, which is also the main focus in Pratt (1964). The primary difficulty in extending the main result to large risks is that there could be a singular equilibrium such that the equilibrium cannot be a function of endowments or uncertainty parameters. However, where a representative agent exists, equilibrium uniquely exists and thus, the small risk results can be extended to large risks. Specifically, with an identical homothetic function, the uniqueness of the equilibrium is guaranteed (see Chipman (1974)) and the local results can be extended to global results. In this case, this paper's main result with a small risk can be extended to large risks. However, with heterogeneous preferences, the uniqueness of the equilibrium is not guaranteed with large risks. Further research should investigate how to measure welfare costs with large riskswhen there is heterogeneity.

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