

The real effects of inflation volatility driven by sunspots

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Abstract

This paper provides a theoretical approach to investigate the real effects of inflation volatility driven by a self-fulfilling prophecy (i.e., sunspot equilibrium) on investment, capital accumulation, and welfare in a representative-agent macroeconomics model. The model is constructed based on empirical evidence that (1) inflation volatility (rather than inflation rates) negatively affects national investment, (2) uncertainty about inflation is attributed more to market psychology than economic fundamentals, and (3) firms are risk-averse decision makers. We show that the introduction of inflation-indexed bonds can improve welfare. A steady-state analysis quantifies the impact of inflation volatility on investment and welfare.

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1. Introduction

Empirical studies have found a negative relationship between inflation volatility and aggregate investment, and the ramifications of inflation volatility have been clearly demonstrated.¹ As such, it is important to ascertain the contributing factors of inflation volatility. If inflation volatility is attributed to intrinsic uncertainty such as technological shocks, this volatility would be the result of market clearing conditions. The corresponding equilibrium allocations would also be efficient based on the second welfare theorem. Most real business cycle models assume that total factor productivity is volatile, so nominal volatility is derived from a price-level adjustment to the productivity change resulting in market efficiency. With this intrinsic uncertainty, any government policy that subdues nominal volatility will not improve welfare. However, if the volatility is the result of extrinsic uncertainty (i.e., sunspots or animal spirits), the volatility-stabilizing policy will improve welfare. Empirical evidence also supports that inflation uncertainty is attributed more to market psychology than fundamentals, which contrasts with the fact that the inflation trend is mainly determined by monetary policy.²

This paper introduces an incomplete-markets macroeconomic model in which price-level volatility is endogenously generated by sunspots. Sunspots are a theoretical device to generate random phenomena that do not affect the fundamentals but may affect the equilibrium outcomes. Sunspot equilibria provide a rational-explanation for the expectations explanation of excess market volatility (see Cass and Shell (1983)). This paper investigates how extrinsic volatility causes inefficiency in a production economy.

As shown in Cass (1992), market incompleteness allows sunspots to affect equilibrium allocation. However, the economy described in this paper is distinct compared to previous sunspot models. The real effect of sunspots in the conventional model is attributed to heterogeneous consumers who, as borrowers and lenders, trade nominal securities. Real returns of nominal securities are affected by excess price-level volatility and, consequently, both borrowers and lenders will have volatile financial income. However, in the economy presented in

¹Grier and Perry (2000) find that monthly inflation volatility reduces output growth contemporaneously over various sample periods, while Davis and Kanago (1996) find that a one-standard deviation shock to inflation decreases real output growth about 1% after six months. Judson and Orphanides (1999) show that countries having experienced high inflation and inflation volatility have a lower rate of economic growth. Cunningham, Tang, and Vilasuso (1997) find that the negative effect of inflation uncertainty has existed only since the mid-1970s. Benhabib and Spiegel (2009) show that there is a non-linear relationship between growth and inflation levels so as to capture an inverted U-shape, which implies high inflation strongly affects lower growth. They also show that in a sample of high inflation episodes, the negative effects of high inflation are associated with inflation volatility rather than the level of inflation.

²Empirical evidence shows that there is a positive relationship between the level and variance of inflation given that the strong trend of inflation amplifies the effects of fundamental fluctuation or increases the level of self-fulfilling inflation uncertainty. See Okun (1971), Kiley (2000; 2007), Grier and Grier (2006).

this paper, there is one representative consumer whose only trading partner is the representative firm.³ In this context, the excess price-level volatility derived from sunspots influences capital trades between the consumer and the firm. When the firm borrows capital, it has expectations about the real value of future interest rates. If the price level is expected to be stable, the real interest rate is also expected to be stable. However, the price-level volatility from sunspots makes the real interest rate volatile, which can affect the firm's investment decisions.

Although sunspots can trigger endogenous price volatility, they would not affect the firm's investment decisions if the firm is risk-neutral. A common assumption in many macroeconomics models is that firms are risk-neutral, but there is considerable evidence that firms' investment decisions are made by risk-averse managers rather than by risk-neutral ones. Risk-averse firms' decisions have been investigated in corporate finance research and under perfect-competitive market conditions (see Baron (1970), Sandmo (1971), and Leland (1972)). In agency problem theory, if firm owners wish to maximize expected profits, the managers may lead the firm to behave in a risk-averse way. The firm's risk-bearing behavior can be observed by corporate hedging activity (Géczy et al. (1997)), and thus risk averse firms will react to inflation uncertainty. Much empirical evidence shows the negative relationship between inflation uncertainty and firm activity. Specifically, inflation volatility is negatively correlated with financial market activity such as the volume of stock market trading and bank lending (Boyd et al. (2001)). Inflation volatility is also related to firms' capital costs and real wages. Thus, any resulting fluctuations of output prices and real wages due to inflation uncertainty would result in lower firm investment (Huizinga (1993)).⁴

Consistent with the empirical evidence, this paper shows that inflation volatility driven by extrinsic uncertainty has a negative impact on national investment, growth, and welfare. In a two-period model with many states, we show that the economy experiences an underinvestment problem when sunspots create inflation volatility. To deal with the real indeterminacy of sunspot equilibria, we use a local analysis around non-sunspot equilibria and conduct comparative statics. By proving that the Hessian matrix of the firm's demand function is negative semi-definite, we show that a sunspot economy has an underinvestment problem. We also show that with two sunspot states, the model can be extended to an infinite period in which a steady-state analysis is available.

³To the best of my knowledge, Kajii (2009) first introduces a production economy in the sunspot equilibrium model. In his model, the firm is risk neutral. The real indeterminacy is generated by heterogeneous consumers' trading in the security markets. He shows that consumers' precautionary motives increase the expected return of securities with sunspots, which causes an overinvestment problem for firms.

⁴Huizinga (1993) also finds that industries with higher fluctuation on resource prices and real wages have lower investment related to their existing capital stock, which indirectly demonstrates firm's risk-bearing investment decisions.

An underinvestment problem provides a justification for sunspot stabilization policies. This paper suggests that introducing inflation-indexed bonds can eliminate the real effect of sunspots. When indexed bonds are introduced in the economy, a firm will want to purchase capital through indexed bonds rather than nominal bonds. Indexation of bonds has been suggested as a stabilizing policy in many previous studies. However, in contrast with previous literature where the goal of stabilizing policies has often been suppressing real volatility, our model aims to increase real investment volume.

We extend the model to an infinite period and analyze steady-state equilibrium. This steady-state analysis shows that the nominal volatility derived from sunspots significantly decreases national investment, resulting in a substantial decrease in aggregate welfare. To derive a steady state, the firm's utility function should be homogeneous of degree one and its domain should be defined on both negative and positive values.⁵ We find a utility function that satisfies both properties – homogeneous degree one and negative domain.⁶ We calibrate the parameter of the firm's utility based on the inflation-uncertainty elasticity to investment. Considerable empirical evidence points to a negative relationship between inflation volatility and investment. In this paper, we calibrate the parameter assuming that 3% of inflation volatility decreases investment by about 1.3%. We also quantify the welfare gain by removing the 3% inflation uncertainty through a sunspot-stabilizing policy. Our simulation shows that the welfare gain is equivalent to an additional 42% consumption in the period when the policy is applied. This quantitative result shows that the impact of inflation uncertainty on welfare is large even with a small value of inflation-uncertainty elasticity of investment.⁷

The rest of the paper is organized as follows. In Section 2, we introduce a representative-agent economy model with sunspots. Section 3 shows that using a local analysis, inflation uncertainty driven by sunspots would cause an underinvestment problem. Section 4 introduces a sunspot-stabilizing policy that can suppress nominal volatility and improve welfare. Section 5 extends the model into infinite periods and conducts a steady-state analysis. Finally, concluding remarks are presented in Section 6.

⁵The firm's economic profits in the presence of nominal volatility can be negative so its utility function should be defined on both negative and positive domains.

⁶Utility functions with constant relative risk aversion are not defined based on a negative domain. Those with constant absolute risk aversion are not homogenous degree one.

⁷That a 3% inflation volatility decrease in investment of about 1.3% implies that the elasticity is 0.45, which is much smaller than Fischer's (2013) calculation. Fischer (2013) showed that a 1% increase in inflation volatility (approximately 0.87 standard deviations of the historical mean), is associated with a 10% reduction in total business investment using administrative loan data from banks in the Dominican Republic from 2001 to 2008.

2. Two-period model

This section provides a two-period general equilibrium model in which there is a representative consumer and a representative firm. The model will be extended to an infinite-period model in section 5. In the first period (period 0), the consumer makes consumption-savings decisions. In the second period (period 1), there are $S < \infty$ sunspot states, $s = 1, \dots, S$. These are indexed underscript s . At the beginning of the second period, the state is publicly observable. State s , $s = 1, \dots, S$ occurs with probability $\mu_s > 0$.

2.1. Representative consumer

The representative consumer's consumption allocation is $c = (c_0, c_1, c_2, \dots, c_s, \dots, c_S) \in \mathbb{R}_{++}^{S+1}$, which corresponds to normalized price $p = (p_0, p_1, p_2, \dots, p_s, \dots, p_S) \in \mathbb{R}_{++}^{S+1}$ where $p_0 = 1$. The consumer's lifetime utility is given by

$$\sum_{s=1}^S \mu_s u(c_0, c_s),$$

where the instantaneous utility function $u(\cdot)$ is strictly increasing, strictly concave, twice-continuously differentiable, and satisfies the von Neumann-Morgenstern expected utility hypothesis. We assume that the closure of indifference curves are contained in \mathbb{R}_{++}^2 .⁸

The consumer is endowed with e_0 units of consumption good in period 0.⁹ In period 1, the consumer is endowed with 1 unit of labor supply that is assumed to be perfectly inelastic. In the general equilibrium, there are some positive spot prices $p \gg 0$ and associated nominal savings (b) such that the consumer chooses (c, b) in the optimization problem:

$$\max \quad \sum_{s=1}^S \mu_s u(c_0, c_s) \tag{1}$$

$$\text{subject to } \begin{cases} p_0 c_0 + b \leq p_0 e_0, \\ p_s c_s \leq (1 + R)b + p_s w_s + p_s \pi_s, \quad s = 1, \dots, S, \end{cases} \tag{2}$$

$$\text{and } c \in \mathbb{R}_{++}^{S+1},$$

where R is the nominal interest rate, w_s is the real wage in state s . π_s is the dividend income (firm's profit) in state s . b represents the nominal bond whose real payoffs are affected by the inflation level in period 1. For the higher (lower) level of inflation in period 1, the real

⁸This can be stated as a limiting conditions, such that $\lim_{c_0 \rightarrow 0} \partial u / \partial c_0 = \infty$, $\lim_{c_1 \rightarrow 0} \partial u / \partial c_1 = \infty$, $\lim_{c_0 \rightarrow 0} \partial u / \partial c_1 = 0$, and $\lim_{c_1 \rightarrow 0} \partial u / \partial c_0 = 0$.

⁹Alternatively, we can assume that the endowment in period 0 (e_0) is produced from period-0 capital (k_0) such that $e_0 = f(k_0) + (1 - d)k_0$.

payoffs of the nominal savings are lower (higher). From the maximization problem of *Eqs.* (1) and (2), the consumer's objective function can be written as

$$\max_b \sum_{s=1}^S \mu_s u(e_0 - b, (1 + r_s)b + w_s + \pi_s), \quad (3)$$

where r_s is the real interest rate in state s , so we have $1 + r_s = (1 + R)/p_s$. We write $\tilde{r} = (r_1, \dots, r_S)$ for the vector of returns.

The objective function in *Eq.* (3) is concave in b . Thus, the consumer's optimal choice is characterized by a solution to the first-order condition,

$$\begin{aligned} - \sum_{s=1}^S \mu_s u_1(e_0 - b, (1 + r_s)b + w_s + \pi_s) + \\ \sum_{s=1}^S \mu_s u_2(e_0 - b, (1 + r_s)b + w_s + \pi_s)(1 + r_s) = 0 \end{aligned} \quad (4)$$

The solution of *Eq.* (4) is unique.

2.2. Representative firm

The representative firm's production function defined on capital (K) and labor (N) is homogeneous of degree one, differentiable strictly increasing, differentiable concave and satisfies the Inada conditions. The production function is denoted as $zF(K, N)$ where z represents the total factor productivity. The per-capita production function is defined as $zf(k)$ where $k = K/N$ and $f(k) = F(K, N)/N$. In the two-period model, we simply assume $z = 1$.

The representative firm determines the amount of real capital borrowing (k) in period 0. The firm's real profit in state s is

$$\pi_s = f(k) - (r_s + d)k - w_s, \quad (5)$$

where r_s is the real interest rate in state s and d is the capital depreciation rate. w_s represents the real wage in state s . Without any intrinsic volatility, there would be no volatility in the real wage that is equal to the marginal product of labor. Therefore, we have $w_s = w_{s'}$ for any $s, s' = 1, \dots, S$ at equilibrium. By the market clearing conditions, the amount of real savings is equal to the real capital level (i.e., $k = b$). Because the period-1 capital amount is determined by the capital market in period 0, for a higher (lower) price level, the real interest is lower (higher). This volatility of the real interest rates makes the firm's profit volatile.

Given $(r_s, w_s)_{s=1}^S$, the firm maximizes the following expected utility of period-1 real profit:

$$\max_k \sum_{s=1}^S \mu_s v(\pi_s) \quad (6)$$

where the instantaneous utility function $v(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing, concave, twice-continuously differentiable, and satisfies the von Neumann-Morgenstern expected utility hypothesis. If the firm's utility function is linear, the inflation volatility does not have any impact on market investment levels nor on the aggregate savings amount. As we shown in the next section, the investment level is not affected by extrinsic uncertainty if the firm is risk neutral nor if there is no nominal fluctuations.

The first-order conditions of the firm's objective function in *Eq.* (6) is

$$\sum_{s=1}^S \mu_s v'(f(k) - (r_s + d)k - w_s) (f'(k) - (r_s + d)) = 0 \quad (7)$$

where the equilibrium wage is

$$w_s = f(k) - kf'(k). \quad (8)$$

From *Eqs.* (7) and (8), we have the following first-order condition:

$$\sum_{s=1}^S \mu_s v'(kf'(k) - (r_s + d)k) (f'(k) - (r_s + d)) = 0 \quad (9)$$

The solution of *Eqs.* (9) is unique based on the concavity of the firm's vNM function.

2.3. Market clearing conditions

In the described economy, we have the following commodity market-clearing conditions:

$$c_0 = e_0 - b, \quad (10)$$

$$c_s = f(k) + (1 - d)k \quad \text{for } s = 1, \dots, S. \quad (11)$$

From the market-clearing conditions of *Eq.* (11), we know that, regardless of the existence of nominal uncertainty, there would be no uncertainty on consumption, i.e., $c_s = c_{s'}$ for all $s, s' = 1, \dots, S$. However, the nominal uncertainty would decrease the real level of investment, which would result in an underinvestment problem.

2.4. Equilibrium

In the analysis, we decompose \tilde{r} into $\tilde{r} = q\tilde{i}$ where q is a scalar and \tilde{i} is a random variable that is co-linear to \tilde{r} . If $E[\tilde{i}] = 1$, q can be interpreted as the expected return of the bond. Then, the consumer's demand for the bond and the firm's demand for capital can be written in terms of $(q, \tilde{i}) \in \mathbb{R}_{++}^{S+1}$ as $B(q, \tilde{i})$ and $K(q, \tilde{i})$, respectively. For a vector of return $\tilde{i} = (i_1, \dots, i_s, \dots, i_S)$ and the expected return of the bond q , $B(q, \tilde{i})$ is the unique solution to *Eqs. (4)* and $K(q, \tilde{i})$ is the unique solution to *Eqs. (9)*.

The return of the bond (q, \tilde{i}) is endogenously determined in the markets. Thus, the rational expectation equilibrium of this economy is defined as follows:

Definition 1. Sunspot equilibrium in the economy with nominal bonds is a vector of returns $\tilde{i} = (i_1, \dots, i_s, \dots, i_S)$ and the expected return q such that

- (i) the bond market clears, i.e., $B(q, \tilde{i}) = K(q, \tilde{i})$.
- (ii) $B(q, \tilde{i})$ maximizes the consumer's objective function in *Eq. (4)*.
- (iii) $K(q, \tilde{i})$ maximizes the firm's objective function in *Eq. (9)*.

If $S = 1$, the model is a standard two-period model of a representative consumer and a representative firm. The economy with $S = 1$ has certainty equilibrium that is Pareto optimal. Assume that $(\bar{q}, 1)$ is the equilibrium return in certainty equilibrium, the return $(\bar{q}, \tilde{1}) \in \mathbb{R}_{++}^{S+1}$ would be one of the equilibria where $S > 1$ and $\tilde{1} = (1, \dots, 1) \in \mathbb{R}_{++}^S$. This equilibrium is called a non-sunspot equilibrium, which is Pareto optimal.

There exists sunspot equilibrium such that the equilibrium return (q, \tilde{i}) is different from non-sunspot equilibrium $(\bar{q}, \tilde{1})$. The model in this paper describes an incomplete-market economy with S number of states and one financial asset. Therefore, around the non-sunspot equilibrium, there exists a continuum of sunspot equilibria with $S - 1$ degrees of freedom.

3. Local analysis

This section analyzes the first-order and the second-order effects of sunspot states on the equilibrium level of capital. The basic techniques of the local analysis are originally introduced by Goenka and Prechac (2006) and Kajii (2007). Goenka and Prechac (2006) uses the analysis to investigate heterogenous sunspot effects on welfare in an endowment economy. Kajii (2007) analyzes the welfare effects of sunspots in the economy with many risk averse consumers and one risk neutral producer. Previous literature on sunspot equilibrium including Goenka and Prechac (2006) and Kajii (2007) has shown that the nominal volatility derived from sunspots triggers real volatility. As a result, the welfare of risk averse consumers

is negatively impacted. This paper displays another mechanism of sunspots that results in welfare being negatively impacted. The nominal volatility from sunspots induces passive investment by risk averse firms, which results in the national investment level falling short of the efficient investment level. In an economy with a representative consumer and firm, nominal volatility does not trigger real volatility but it can affect real investment decisions, as reflected in this section.

3.1. Basic setting

We are interested in the set of utility profiles around a non-sunspot equilibrium. For any $S > 1$, generically in endowments, there is finite number of non-sunspot equilibria. For any non-sunspot equilibrium $(\bar{q}, \bar{1}) \in \mathbb{R}_{++}^{S+1}$, we have $B(q, \tilde{i}) - K(q, \tilde{i}) \neq 0$. Write \tilde{i}_{-S} as (i_1, \dots, i_{S-1}) , and set $i_S = \left(1 - \sum_{s=1}^{S-1} \mu_s i_s\right) / \mu_S$ for the normalization, $E[\tilde{i}] = 1$. Using this convention, define $\hat{B}(q, \tilde{i})$ and $\hat{K}(q, \tilde{i})$ by the following rule:

$$\hat{B}(q, \tilde{i}) = B\left(q, \left(\tilde{i}_{-S}, \left(1 - \sum_{s=1}^{S-1} \mu_s i_s\right) / \mu_S\right)\right) \quad (12)$$

and

$$\hat{K}(q, \tilde{i}) = k\left(q, \left(\tilde{i}_{-S}, \left(1 - \sum_{s=1}^{S-1} \mu_s i_s\right) / \mu_S\right)\right) \quad (13)$$

Then, our task is to find the derivatives of \hat{B} and \hat{K} with respect to \tilde{i}_{-S} and evaluate them at $\tilde{i}_{-S} = \bar{1}_{-S}$.

By checking the Hessian matrix of \hat{K} and \hat{B} , this section shows that non-sunspot equilibrium constitutes a local maximum, which implies that sunspot equilibria results in an underinvestment problem.

3.2. Sunspot effects on the firm

This subsection shows that the Hessian matrix of \hat{K} is negative semi-definite if the firm is risk averse. This implies that its demand for capital decreases by the nominal uncertainty generated by sunspots.

To describe the corresponding first-order condition of the firm in Eq. (9), we set

$$F_f(q, \tilde{i}, k) := \sum_{s=1}^S \mu_s v'(k f'(k) - (q i_s + d) k) (f'(k) - (q i_s + d)) = 0. \quad (14)$$

The first-order condition can be written as $F_f(q, \tilde{i}, k) = 0$.

Our task is to find the derivatives of \hat{K} with respect to \tilde{i}_{-S} and evaluate them at $\tilde{i}_{-S} = \bar{1}_{-S}$. The following lemma shows that around non-sunspot equilibrium, the first-order effect

on $\widehat{K}(q, i_{-s})$ is null but the second-order effect is not null if the firm is risk averse.

Lemma 1 *At the non-sunspot equilibrium bond return $(\bar{q}, \bar{1})$,*

$$\frac{\partial \widehat{K}}{\partial i_s} = 0, \quad (15)$$

for every $s = 1, \dots, S - 1$, and

$$\left(\frac{\partial^2}{\partial i_s \partial i_{s'}} \right)_{s, s'} \widehat{K} \quad (16)$$

is a negative definite matrix if $v(\cdot)$ is strictly concave.

Proof. From the first-order condition in Eq. (14), we have

$$F_f(q, \tilde{i}, k) := \sum_{s=1}^S \mu_s v'(k f'(k) - (q i_s + d)k) (f'(k) - (q i_s + d)) = 0. \quad (17)$$

To keep the normalization $E[\tilde{i}] = 1$, we define $G(q, \tilde{i}^{-S}, k)$ by the rule:

$$G(q, \tilde{i}_{-S}, k) = F_f \left(q, \left(\tilde{i}_{-S}, \left(1 - \sum_{s=1}^{S-1} \mu_s i_s \right) / \mu_S \right), k \right). \quad (18)$$

First, we show that $\partial \widehat{K} / \partial i_s = 0$ at $\tilde{i} = \bar{1}$. Implicitly differentiating $G(q, \tilde{i}^{-S}, k)$ with respect to i_s , we have

$$\frac{\partial}{\partial i_s} G = \frac{\partial}{\partial i_s} F_f - \left(\frac{\mu_s}{\mu_S} \right) \frac{\partial}{\partial i_S} F_f \quad \text{for } s = 1, \dots, S - 1 \quad (19)$$

and

$$\frac{\partial}{\partial k} G = \frac{\partial}{\partial k} F_f. \quad (20)$$

Thus, differentiating $G(q, \tilde{i}_{-S}, k) = 0$ with i_s and using Eqs. (19) and (20), we have

$$\begin{aligned} & \frac{\partial}{\partial i_s} G + \frac{\partial}{\partial i_s} \widehat{K} \frac{\partial}{\partial k} G \\ &= \frac{\partial}{\partial i_s} F_f - \left(\frac{\mu_s}{\mu_S} \right) \frac{\partial}{\partial i_S} F_f + \frac{\partial}{\partial i_s} \widehat{K} \frac{\partial}{\partial k} F_f = 0. \end{aligned} \quad (21)$$

By the symmetry relation, we have

$$\frac{1}{\mu_s} \frac{\partial}{\partial i_s} F_f = \frac{1}{\mu_{s'}} \frac{\partial}{\partial i_{s'}} F_f, \quad (22)$$

for all $s, s' = 1, \dots, S$.

From Eqs. (21) and (22), we have

$$\frac{\partial}{\partial i_s} \widehat{K} \frac{\partial}{\partial k} F_f = 0. \quad (23)$$

Because $\frac{\partial}{\partial k} F \neq 0$, we have

$$\frac{\partial}{\partial i_s} \widehat{K} = 0. \quad (24)$$

Next, we compute $\frac{\partial^2 \widehat{K}}{\partial i_s \partial i_{s'}}$. Set

$$\gamma = -\frac{\partial}{\partial k} F_f = -\frac{\partial}{\partial k} G > 0$$

and

$$\alpha = \frac{1}{\mu_s} \frac{\partial^2 F_f}{\partial (i_s)^2} = \frac{1}{\mu_{s'}} \frac{\partial^2 F_f}{\partial (i_{s'})^2}.$$

Implicitly differentiate G with respect to i_s , we have

$$\frac{\partial G}{\partial i_s} + \frac{\partial \widehat{K}}{\partial i_s} \frac{\partial G}{\partial k} = 0. \quad (25)$$

Implicitly differentiate Eq. (25) with respect to $i_{s'}$, we have

$$\frac{\partial^2 G}{\partial i_s \partial i_{s'}} + \frac{\partial \widehat{K}}{\partial i_{s'}} \frac{\partial^2 G}{\partial i_s \partial k} + \frac{\partial^2 \widehat{K}}{\partial i_s \partial i_{s'}} \frac{\partial G}{\partial k} + \frac{\partial \widehat{K}}{\partial i_s} \frac{\partial \widehat{K}}{\partial i_{s'}} \frac{\partial G}{\partial k \partial i_{s'}} = 0. \quad (26)$$

Because $\frac{\partial \widehat{K}}{\partial i_{s'}} = 0$ when $\tilde{i}_{-S} = \tilde{1}_{-S}$, Eq. (26) is

$$\frac{\partial^2 G}{\partial i_s \partial i_{s'}} + \frac{\partial^2 \widehat{K}}{\partial i_s \partial i_{s'}} \frac{\partial G}{\partial k} = 0, \quad (27)$$

which, in turn, is equivalent

$$\frac{\partial^2 \widehat{K}}{\partial i_s \partial i_{s'}} = - \left(\frac{\partial G}{\partial k} \right)^{-1} \frac{\partial^2 G}{\partial i_s \partial i_{s'}} = \frac{1}{\gamma} \frac{\partial^2 G}{\partial i_s \partial i_{s'}}. \quad (28)$$

Now, we need to prove that $\frac{\partial^2 G}{\partial i_s \partial i_{s'}}$ is positive definite and γ is strictly negative if $v(\cdot)$ is

strictly concave. We have

$$\begin{aligned}
\frac{\partial^2 G}{\partial i_s \partial i_{s'}} &= \frac{\partial}{\partial i_{s'}} \left(\frac{\partial}{\partial i_s} F_f - \left(\frac{\mu_s}{\mu_S} \right) \frac{\partial}{\partial i_S} F_f \right) \\
&= \left(\frac{\partial^2}{\partial i_{s'} \partial i_s} F_f - \left(\frac{\mu_{s'}}{\mu_S} \right) \frac{\partial^2}{\partial i_s \partial i_S} F_f \right) \\
&\quad - \left(\frac{\mu_s}{\mu_S} \right) \left(\frac{\partial^2}{\partial i_{s'} \partial i_S} F_f - \left(\frac{\mu_{s'}}{\mu_S} \right) \frac{\partial^2}{\partial i_S^2} F_f \right).
\end{aligned} \tag{29}$$

Because from Eq. (17), we have

$$\frac{\partial^2}{\partial i_s \partial i_S} F_f = 0 \text{ for } s = 1, \dots, S-1, \tag{30}$$

we can simplify Eq. (29) as

$$\begin{aligned}
\frac{\partial^2 G}{\partial i_s \partial i_{s'}} &= \frac{\partial^2}{\partial i_{s'} \partial i_s} F_f + \left(\frac{\mu_s \mu_{s'}}{\mu_S \mu_S} \right) \frac{\partial^2}{\partial i_S^2} F_f \\
&= \frac{\partial^2}{\partial i_{s'} \partial i_s} F_f + \left(\frac{\mu_s \mu_{s'}}{\mu_S} \right) \alpha.
\end{aligned} \tag{31}$$

From Eqs. (30) and (31), we have

$$\frac{\partial^2 G}{\partial i_s \partial i_{s'}} = \begin{cases} \left(\frac{\mu_s \mu_{s'}}{\mu_S} \right) \alpha & \text{if } s \neq s' \\ \mu_s \alpha + \left(\frac{\mu_s \mu_s}{\mu_S} \right) \alpha & \text{if } s = s' \end{cases}, \tag{32}$$

Therefore, From Eqs. (27) and (32), we have

$$\frac{\partial^2 \widehat{K}}{\partial i_s \partial i_{s'}} = \frac{\alpha}{\gamma} \left(\begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_{S-1} \end{bmatrix} + \frac{1}{\mu_S} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_{S-1} \end{bmatrix} \begin{bmatrix} \mu_1 & \cdots & \mu_{S-1} \end{bmatrix} \right), \tag{33}$$

The matrix inside parenthesis in Eq. (33) is positive definite.

Next, we prove that α is strictly negative if $v(\cdot)$ is strictly concave. Remembering Eq. (17), we have

$$F_f(q, \tilde{i}, k) := \sum_{s=1}^S \mu_s v'(k f'(k) - (q i_s + d) k) (f'(k) - (q i_s + d)) = 0.$$

Then, we have

$$\begin{aligned} \frac{\partial F_f}{\partial i_s} &= -\mu_s q k (f'(k) - (qi_s + d)) v'' (kf'(k) - (qi_s + d) k) \\ &\quad - \mu_s q v' (kf'(k) - (qi_s + d) k) \end{aligned} \quad (34)$$

and

$$\begin{aligned} \frac{\partial^2 F_f}{\partial (i_s)^2} &= \mu_s (qk)^2 (f'(k) - (qi_s + d)) v''' (kf'(k) - (qi_s + d) k) \\ &\quad + 2\mu_s q^2 k v'' (kf'(k) - (qi_s + d) k) \end{aligned} \quad (35)$$

From Eq. (34), we have

$$\begin{aligned} \alpha &= \frac{1}{\mu_s} \frac{\partial^2 F_f}{\partial (i_s)^2} \\ &= (qk)^2 (f'(k) - (qi_s + d)) v''' (kf'(k) - (qi_s + d) k) \\ &\quad + 2q^2 k v'' (kf'(k) - (qi_s + d) k) \end{aligned} \quad (36)$$

At non-sunspot equilibrium (i.e., $r_s = r_{s'}$ for $s, s' = 1, \dots, S$), we have $(f'(k) - (r_s + d)) = 0$ and $\pi_s = 0$ from the first-order condition in Eq. (7). Thus, Eq. (36) can be simplified as

$$\alpha = 2q^2 k v'' (kf'(k) - (qi_s + d) k),$$

which is strictly negative if $v'' < 0$. Therefore, the sufficient and necessary condition for $\frac{\partial^2 \widehat{K}}{\partial i_s \partial i_{s'}}$ to be strictly negative definite is $v'' < 0$. ■

3.3. Sunspot effects on the consumer

The negativity of the Hessian matrix of \widehat{K} in Lemma 1 implies that if the firm is risk averse, its demand for capital decreases by nominal uncertainty. Next, we investigate how the supply of capital, determined by the representative consumer, is affected by extrinsic uncertainty. The nominal volatility driven by extrinsic randomness does not generate real volatility. Therefore, the Hessian matrix of the consumer demand for bond, \widehat{B} , is zero matrix, as shown in the following lemma:

Lemma 2 At $(\bar{q}, \bar{1})$,

$$\frac{\partial \widehat{B}}{\partial i_s} = 0,$$

for every $s = 1, \dots, S - 1$, and

$$\left(\frac{\partial^2}{\partial i_s \partial i_{s'}} \right)_{s,s'} \widehat{B} = 0_{S-1, S-1}.$$

Proof. To describe the corresponding first-order condition of the firm, we set

$$\begin{aligned} F_c(q, \tilde{i}, b) &: = - \sum_{s=1}^S \mu_s u_1 (e_0 - b, (1 + qi_s) b + w_s + \pi_s) + \\ &+ \sum_{s=1}^S \mu_s u_2 (e_0 - b, (1 + qi_s) b + w_s + \pi_s) (1 + qi_s). \end{aligned} \quad (37)$$

Then the first-order condition can be written as $F_c(q, \tilde{i}, k) = 0$. To keep the normalization $E[\tilde{i}] = 1$, we define $E(q, \tilde{i}^{-S}, b)$ by the rule

$$E(q, \tilde{i}^{-S}, b) = F_c \left(q, \left(\tilde{i}_{-S}, \left(1 - \sum_{s=1}^{S-1} \mu_s i_s \right) / \mu_S \right), b \right).$$

First, we show that $\frac{\partial \widehat{B}}{\partial i_s} = 0$ at $\tilde{i} = \tilde{1}$. Implicitly differentiating $E(q, \tilde{i}^{-S}, b)$ with respect to i_s , we have

$$\frac{\partial}{\partial i_s} E = \frac{\partial}{\partial i_s} F_c - \left(\frac{\mu_s}{\mu_S} \right) \frac{\partial}{\partial i_S} F_c \quad \text{for } s = 1, \dots, S - 1 \quad (38)$$

and

$$\frac{\partial}{\partial b} E = \frac{\partial}{\partial b} F_c. \quad (39)$$

Thus, differentiating $E(q, \tilde{i}_{-S}, b) = 0$ with i_s and using Eqs. (38) and (39) we have

$$\begin{aligned} &\frac{\partial}{\partial i_s} E + \frac{\partial}{\partial i_s} \widehat{B} \frac{\partial}{\partial b} E \\ &= \frac{\partial}{\partial i_s} F_c - \left(\frac{\mu_s}{\mu_S} \right) \frac{\partial}{\partial i_S} F_c + \frac{\partial}{\partial i_s} \widehat{B} \frac{\partial}{\partial b} F_c = 0 \end{aligned} \quad (40)$$

By the symmetry relation, we have

$$\frac{1}{\mu_s} \frac{\partial}{\partial i_s} F_c = \frac{1}{\mu_{s'}} \frac{\partial}{\partial i_{s'}} F_c. \quad (41)$$

From Eqs. (40) and (41), we have

$$\frac{\partial}{\partial i_s} \widehat{B} \frac{\partial}{\partial b} F_c = 0. \quad (42)$$

Because $\frac{\partial}{\partial b} F_c \neq 0$, we have

$$\frac{\partial}{\partial i_s} \widehat{B} = 0. \quad (43)$$

In the same way as in the proof of Lemma 1, we can get the Hessian matrix of \widehat{B} :

$$\left(\frac{\partial^2}{\partial i_s \partial i_{s'}} \right)_{s,s'} \widehat{B} = \frac{1}{\bar{\gamma}} \frac{\partial^2 E}{\partial i_s \partial i_{s'}} = \begin{cases} \left(\frac{\mu_s \mu_{s'}}{\mu_S} \right) \bar{\alpha} & \text{if } s \neq s' \\ \mu_s \bar{\alpha} + \left(\frac{\mu_s \mu_s}{\mu_S} \right) \bar{\alpha} & \text{if } s = s' \end{cases}, \quad (44)$$

where

$$\bar{\gamma} = -\frac{\partial}{\partial b} E > 0$$

and

$$\bar{\alpha} = \frac{1}{\mu_s} \frac{\partial^2 F_c}{\partial (i_s)^2}.$$

The remaining proof is to show that

$$\bar{\alpha} = \frac{1}{\mu_s} \frac{\partial^2 F_c}{\partial (i_s)^2} = 0.$$

Remembering Eq. (37), we have

$$F_c(q, \tilde{i}, b) := -\sum_{s=1}^S \mu_s u_1(c_1, c_s) + \sum_{s=1}^S \mu_s u_2(c_1, c_s) (1 + qi_s).$$

where

$$c_0 = e_0 - b$$

and

$$c_s = (1 + qi_s) b + kf'(k) - (qi_s + d) k.$$

Differentiating F_c with i_s at non-sunspot equilibrium, we have

$$\begin{aligned} \frac{\partial F_c}{\partial i_s} &= -\mu_s q b u_{12}(c_1, c_s) + \mu_s q b u_{22}(c_1, c_s) (1 + qi_s) \\ &\quad + \mu_s q u_2(c_1, c_s) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 F_c}{\partial i_s^2} &= -\mu_s q^2 b (b - k) u_{122}(c_1, c_{ss}) + \mu_s q^2 b (b - k) u_{222}(c_1, c_s) (1 + qi_s) \\ &\quad + \mu_s q^2 (b - k) u_{22}(c_1, c_s), \end{aligned} \quad (45)$$

At equilibrium, we have $b = k$. Thus, from Eq. (45), we have

$$\bar{\alpha} = \frac{1}{\mu_s} \frac{\partial^2 F_c}{\partial (i_s)^2} = 0.$$

■

3.4. Underinvestment problem

In Lemma 1, we show that the Hessian matrix of the firm's demand for capital is negative definite if the firm is risk averse. On the other hand, Lemma 2 indicates that the Hessian matrix of the supply of bonds is zero because sunspots do not generate real volatility in this model. The next step is to investigate the general-equilibrium effect of sunspots. Although sunspots do not generate real volatility, they lead to nominal volatility and underinvestment problems, which is shown in the following proposition:

Proposition 1 *If the firm is risk averse, in any sunspot equilibrium close enough to the certainty equilibrium, the economy has an underinvestment problem. If the firm is risk neutral, in any sunspot equilibrium close enough to the certainty equilibrium, the economy has an efficient level of production.*

Proof. Let \bar{k} be the capital level of non-sunspot equilibrium. Applying Lemma 1, \widehat{K} is locally maximized at $\tilde{i}_{-S} = \tilde{1}_{-S}$ if the firm is risk averse. This implies that for any \tilde{i} with $E[\tilde{i}] = 1$ which is close enough to $\tilde{1}$, we have $\widehat{K}(\bar{q}, \tilde{i}) < \bar{k}$. Applying Lemma 2, we have $B(\bar{q}, \tilde{i}) = \bar{k}$ for any \tilde{i} with $E[\tilde{i}] = 1$ which is close enough to $\tilde{1}$. We have $B - K = 0$ in equilibrium, $B(q, \tilde{i})$ is increasing in q , and $K(q, \tilde{i})$ is decreasing in q . Thus, if the firm is risk averse (i.e., $v'' < 0$), there exists unique q' satisfying $q' < q$, $B(q', \tilde{i}) - K(q', \tilde{i}) = 0$, and $K(q', \tilde{i}) < \bar{k}$. This implies that the economy has an underinvestment problem.

If the firm is risk neutral, by Lemmas 1 and 2, we have $B(\bar{q}, \tilde{i}) = K(\bar{q}, \tilde{i}) = \bar{k}$ for any \tilde{i} with $E[\tilde{i}] = 1$ which is close enough to $\tilde{1}$. This implies that the investment level of sunspot equilibrium is the same as that of the certainty equilibrium. ■

Proposition 1 shows that if the firm is risk averse, the economy has an underinvestment problem with sunspots. Underinvestment means that the equilibrium investment level with sunspots is lower than that without sunspots. As the non-sunspot equilibrium allocation is Pareto optimal, any deviation from the non-sunspot equilibrium investment level would violate Pareto optimality. Therefore, we have the following corollary:

Corollary 1 *If the firm is risk averse, in any sunspot equilibrium close enough to the certainty equilibrium, the consumer's welfare is lower than that of non-sunspot equilibrium.*

Proof. The equilibrium allocation with certainty equilibrium is Pareto optimal. Proposition 1 shows that if the firm is risk averse, the equilibrium capital level in any sunspot equilibrium

is different from the level in the certainty equilibrium. This implies that sunspot equilibrium is not Pareto optimal. Because there is a single representative agent in the model, Pareto inefficiency implies Pareto inferiority. ■

We have shown that there exists a continuum of equilibria around certainty equilibrium. Except for certainty equilibrium, the price levels of all equilibria are volatile. Although this volatility does not originate from economic fundamentals, it can affect the real investment plans of risk averse firms. In a conventional macroeconomic model, the firm is a risk-neutral decision maker. This assumption stems from the basis that investors can neutralize the risk of its investment portfolio by diversifying its investment sources. However, even though the investors are risk neutral, evidence shows that managers who decide on their firm's investment decisions are often risk averse.

The risk-averse firm attempts to reduce profit volatility. A higher level of investment can induce higher profit volatility in the presence of inflation uncertainty. Therefore, if the firm expects a higher level of price volatility, it will reduce its future investment. This section shows that when the firm maximizes the expected concave utility of profits, the economy experiences underinvestment problems.

An underinvestment problem is defined as a marginal increase of investment from equilibrium can Pareto-improve the economy. Proposition 1 shows that the investment level of sunspot equilibrium is lower than that of non-sunspot equilibrium. As non-sunspot equilibrium is Pareto optimal, we can conclude that sunspot equilibrium reflects Pareto inferiority in the form of underinvestment.

4. Stabilizing policy through inflation-indexed bonds

The finding that sunspots can cause underinvestment problems provides a justification for sunspot-stabilizing policies.¹⁰ A common stabilizing policy is the indexation of the securities. When inflation-indexed bonds are introduced in the economy, the firm would prefer financing capital through indexed bonds rather than nominal bonds. With risk-free indexed bonds, the risk-averse firm will be reluctant to purchase the risky bonds and, consequently, risky bonds market will be inactive.

With the introduction of indexed bonds, the consumer's budget constraint should be

¹⁰For sunspot-stabilizing policies, see Cass and Shell (1983) and Balasko (1983), Mas-Colell (1992), Antinolfi and Keister (1998), and Kajii (1997). Goenka and Préhac (2006) and Kang (2015, 2019) show that even though stabilizing policies can suppress nominal volatility, they do not necessarily Pareto-improve the economy in a sunspot-equilibrium model with heterogeneous consumers.

$$\begin{aligned}
c_0 + b + x &\leq e_0 \\
p_s c_s &\leq (1 + R)b + p_s w_s + p_s \pi_s + (1 + h)p_s x, \\
&\text{for } s = 1, \dots, S,
\end{aligned} \tag{46}$$

where x is the amount of indexed bonds and h is the real interest of indexed bonds. The firm finances capital through both indexed and nominal bonds. Let k_1 and k_2 be the capital amounts financed by nominal (b) and indexed (x) bonds, respectively. Then, the firm's maximization problem is

$$\max_{k_1, k_2} \sum_{s=1}^S \mu_s V(f(k_1 + k_2) - r_s k_1 - h k_2 - d(k_1 + k_2) - w_s) \tag{47}$$

Under the asset market clearing condition, in equilibrium we have $k_1 = b$ and $k_2 = x$. With the introduction of indexed bonds, we show that $k_1 = 0$ and $k_2 > 0$ in equilibrium, as shown in the following proposition:

Proposition 2 *The introduction of inflation-indexed bonds eliminates the real effect of sunspots and, thus, induces equilibrium allocations to be Pareto-optimal.*

Proof. By contradiction, assume that both indexed bonds and nominal bonds are active. Then, from the first-order conditions of the consumer's maximization problem, we have

$$\sum_{s=1}^S \mu_s u_2(c_0, c_s) (1 + r_s) = \sum_{s=1}^S \mu_s u_2(c_0, c_s) (1 + h). \tag{48}$$

Because $c_s = c_{s'}$ for all $s, s' = 1, \dots, S$, Eq. (48) implies that

$$\sum_{s=1}^S \mu_s r_s = h. \tag{49}$$

The firm's maximization problem is

$$\max_{k_1, k_2} \sum_{s=1}^S \mu_s v(f(k_1 + k_2) - r_s k_1 - h k_2 - d(k_1 + k_2) - w_s).$$

In equilibrium, the firm's marginal utility of k_1 is

$$\sum_{s=1}^S \mu_s v'(\pi_s) (f'(k) - (r_s + d)). \tag{50}$$

In equilibrium, the marginal firm utility of k_2 is

$$\sum_{s=1}^S \mu_s v'(\pi_s) (f'(k) - (h + d)). \tag{51}$$

If the two marginal utilities in *Eqs.* (50-51) are the same, the firm has an incentive to purchase both nominal and indexed bonds. Because $v'(\pi_s)$ is a decreasing function, π_s and $(f'(k) - (r_s + d))$ are negatively correlated, for any k_1 and k_2 , we have

$$\sum_{s=1}^S \mu_s v'(\pi_s) (f'(k) - (r_s + d)) < \sum_{s=1}^S \mu_s v'(\pi_s) (f'(k) - (h + d)),$$

which implies that the firm has no incentive to hold nominal bonds. This contradicts the assumption that both indexed bonds and nominal bonds are active. Therefore, nominal bonds cannot be active in the economy. ■

5. Steady-state analysis

Our model with the representative firm and consumer can be extended to an infinite-period model. In an infinite period with two sunspot states, this section analyses the impact of nominal volatility driven by sunspots on capital accumulation at the steady state. The welfare gains as a result of the sunspot-stabilizing policies will also be quantified.

5.1. Basic setting

We assume that there are two sunspot states ($s = \alpha$ or β) in each period. The representative consumer's budget constraint in year t is

$$\begin{aligned} p_{t,\alpha} c_{t,\alpha} + b_{t+1} &\leq (1 + R_t) b_t + p_{t,\alpha} w_{t,\alpha} + p_{t,\alpha} \pi_{t,\alpha} & \text{if } s = \alpha, \\ p_{t,\beta} c_{t,\beta} + b_{t+1} &\leq (1 + R_t) b_t + p_{t,\beta} w_{t,\beta} + p_{t,\beta} \pi_{t,\beta} & \text{if } s = \beta. \end{aligned}$$

where b_t is the amount of nominal bonds, R_t is the real interest rate, $w_{t,s}$ is the real wage, and $\pi_{t,s}$ is the dividend income in period t and state $s = \alpha, \beta$. Because nominal volatility (i.e., $p_{t,\alpha} \neq p_{t,\beta}$) does not generate real volatility in the model, we can define c_t as the same as $c_{t,\alpha}$ ($= c_{t,\beta}$). Then, the consumer's lifetime utility in period t is defined as

$$U^{(t)}(c_0, c_1, \dots, c_\infty) = \sum_{i=0}^{\infty} \delta^i u(c_{t+i}), \quad (52)$$

where $\delta \in (0, 1)$ is the discount factor. We assume $u(c)$ is the CES instantaneous utility function:

$$u(c) = \begin{cases} \frac{c^{1-\rho}-1}{1-\rho} & \text{if } \rho > 1, \\ \ln c & \text{if } \rho = 1. \end{cases}$$

where ρ is the inverse elasticity of intertemporal substitution.

We consider a standard Cobb-Douglas production function where K_t , N_t , and z_t represent aggregate capital, aggregate labor, and exogenous productivity, respectively:

$$Y_t = z_t K_t^a N_t^{1-a}. \quad (53)$$

where $a \in (0, 1)$ represents the capital share of output. Because we also assume that the representative consumer inelastically supplies one unit of labor, the labor supply N_t would be equal to one in equilibrium. We assume that the firm maximizes the expected utility of profit in each period. The firm's expected utility function $v(\pi)$ is

$$v(\pi) = \begin{cases} \pi & \text{if } \pi > 0 \\ \theta\pi & \text{if } \pi < 0 \end{cases}, \quad (54)$$

where $\theta > 1$ is the risk aversion parameter. The main reasons for using the non-differentiable aversion function for the firm's vNM function are (1) its domain includes a negative value (when there are negative economic profits) and (2) it is a homogenous-degree-one function. With commonly used risk preferences such as CRRA or CARA, we cannot derive steady-state equilibrium. CRRA utility functions are not defined on the negative profits. CRRA utility functions are not a homogeneous degree of one.

Although the risk aversion function in Eq. (54) is not differentiable at $\pi = 0$, equilibrium is guaranteed to exist in the two-state model. The economic profits with Cobb-Douglas production function is zero without nominal volatility so the economic profits would fluctuate around zero with nominal volatility. In the two-states model, the profit in one state is strictly positive and that in the other state is strictly negative. Therefore, this risk-aversion function captures the firm's risk reluctance to nominal volatility.

5.2. Price-level volatility

Decomposing the price level \tilde{P} into its expected value and volatility, we have

$$\tilde{P} = P + P\tilde{q} \quad \text{where} \quad E[\tilde{q}] = 0. \quad (55)$$

In Eq. (55), the relative standard deviation of \tilde{P} is equal to the standard deviation of \tilde{q} . We remove the subscript t in all nominal variables because we assume that in each state, extrinsic random variables are independent and identically distributed across time. Our goal in this section is to investigate how the volatility \tilde{q} (as a measure of relative standard deviation) affects the investment level and welfare at a steady state. Once \tilde{q} is fixed, the expected price level (P) is determined in equilibrium. We can also parameterize the real interest in terms

of relative volatility, that is

$$\tilde{r} = r + r\tilde{x} \quad \text{where} \quad E[\tilde{x}] = 0. \quad (56)$$

In equilibrium, we have

$$(1 + \tilde{r}) = \frac{1 + R}{\tilde{P}}. \quad (57)$$

From *Eqs.* (55-57) we have

$$(1 + R) = \frac{(1 + r)P}{E\left[\frac{1}{1+\tilde{q}}\right]}. \quad (58)$$

From *Eqs.* (56-58), we can solve for \tilde{x} in terms of r and \tilde{q} .

$$\tilde{x} = \left(\frac{1+r}{r}\right) \left\{ \frac{1}{(1+\tilde{q})E\left[\frac{1}{1+\tilde{q}}\right]} - 1 \right\} \quad (59)$$

If the probabilities of the two states are equal (i.e., $\mu_\alpha = \mu_\beta = 1/2$) such as

$$\tilde{q} = \begin{cases} \sigma & \text{with probability of } 1/2 \quad (\text{state } \alpha) \\ -\sigma & \text{with probability of } 1/2 \quad (\text{state } \beta) \end{cases}, \quad (60)$$

the relative standard deviation of \tilde{P} would be σ . Assuming that the price level is high at state α and low at state β as shown in *Eq.* (60), the real interest rates from *Eqs.* (56), (59), and (60) should be

$$r_\alpha = r - (1+r)\sigma \quad \text{and} \quad r_\beta = r + (1+r)\sigma. \quad (61)$$

5.3. Steady-state equilibrium

In the rational expectation model, the consumer can perfectly estimate not only the future interest rates but also future capital, labor, and dividend incomes. In the economy, the Euler equation is characterized as

$$u'(c_t) = E(\tilde{r}_{t+1} + 1 - d) \delta u'(c_{t+1}), \quad (62)$$

Because there is no consumption volatility, the Euler equation in *Eq.* (62) can be written as

$$u'(c_t) = (r_{t+1} + 1 - d) \delta u'(c_{t+1}), \quad (63)$$

where

$$\tilde{r}_{t+1} = r_{t+1} + r_{t+1}\tilde{x}_{t+1}.$$

The total factor productivity (z_t) is assumed to grow exogenously at the rate g_z . Therefore, in the steady state, capital and output must grow at the rate $g_z/(1 - \alpha) \equiv g$. Then, the steady-state condition from Eq. (63) is

$$\delta^{\frac{1}{\rho}} (r_t + 1 - d)^{\frac{1}{\rho}} = \exp(g) \quad (64)$$

The Euler equation from the firm's maximization is

$$Ev'(\tilde{\pi}_t) (f'_t(k_t) - (\tilde{r}_t + d)) = 0 \quad (65)$$

which is equivalent to

$$f'_t(k_t) - (r_{t,\alpha} + d) + \theta \{f'_t(k_t) - (r_{t,\beta} + d)\} = 0 \quad (66)$$

where

$$f_t(k_t) = z_t k_t^a \text{ and } f'_t(k_t) = a z_t k_t^{a-1}.$$

With the parameter choices of $a = 0.36$, $d = 0.08$, $g = 0.02$, $\delta = 0.981$, $\rho = 1$, and when the economy has no price volatility (i.e., $\sigma = 0$), the capital-to-output ratio ($k_t/y_t = K_t/Y_t$) at the steady state is 3.

From Eqs. (54), (61), (64), and (66), we can derive the steady-state capital-to-output ratio in terms of the relative standard deviation of inflation volatility (σ) and the risk-aversion parameter (θ) as shown in Table 1.¹¹

		Inflation volatility (σ)			
		0%	1%	3%	5%
Risk-aversion parameter (θ)	1	3.00	3.00	3.00	3.00
	2	3.00	2.916	2.761	2.621
	3	3.00	2.875	2.655	2.466
	4	3.00	2.852	2.595	2.381
	5	3.00	2.836	2.557	2.328

Table 1. Steady-state capital-to-output ratio varying the relative standard deviation of inflation volatility (σ) and the risk-aversion parameter (θ)

¹¹There have been many empirical attempts to measure the absolute risk aversion of firms at the micro level. Babcock, Choi and Feinerman (1993) show that the measure of absolute risk aversion can vary with firm size and uncertainty size. To the best of our knowledge, no research has been conducted on the empirical measure of CARA in the macroeconomic firm-level.

The steady-state investment rate (I/Y) is computed as

$$I/Y = (K/Y) (\exp(g) - (1 - d)). \quad (67)$$

Where $g = 0.02$ and $d = 0.08$, *Eq. (67)* become $I/Y = 0.1002 (K/Y)$. This implies that a 10% decrease in the capital-to-output ratio results in a 1% decrease in the investment rate, which results in a $(1 - \alpha) \times 1\%$ decrease in the GDP in the next period. Therefore, we can calibrate the value of the risk-aversion parameter (θ) based on investment (or GDP growth) elasticity of inflation volatility.

Several studies have investigated the impact of inflation uncertainty on output growth or investment. Elder (2004) shows that the inflation-uncertainty elasticity of output growth was around 1.66 in the U.S. from 1982 to 2000. Fischer (2013) also shows that a 1% increase in inflation volatility (approximately 0.87 standard deviations of the historical mean), is associated with a 10% reduction in total business investment. Although there are many different channels for inflation uncertainty to decrease output, in this paper we assume that the decreased investment due to the firm's risk aversion is the main channel, which is also assumed in Elder (2004). The value of 10 as the inflation-volatility elasticity of investment shown in Fischer (2013) is very large in our model. To achieve this elasticity value, the value of θ should be more than 50, which is unreasonably large. In this paper, we assume that the inflation-volatility elasticity of investment is around 0.5 in which the loss averse parameter (θ) is calibrated at around 4.¹² Even with this small value of inflation-volatility elasticity, this paper shows that the welfare loss from inflation uncertainty is significant.

5.4. Steady-state transition and welfare analysis

For a welfare analysis, we need to simulate the equilibrium transition from the steady-state with inflation volatility ($\sigma > 0$) to that without volatility ($\sigma = 0$). In Section 5, we show that the stabilizing policy can eliminate the volatility (i.e., $\sigma = 0$). The numerical methodology for finding the convergence path when stabilizing policy is applied in year 1

¹²Specifically, $\sigma = 1\%$, the capital-output ratio is 2.852 if $\theta = 4$. In this case, the inflation-uncertainty elasticity of investment is around -0.5 because

$$\frac{3 - 2.852}{3} \times \underbrace{0.1}_{=\exp(g)-(1-d)} = 0.49\%.$$

When $\sigma = 3\%$ and $\theta = 4$, the elasticity is around 0.45 since

$$\frac{3 - 2.595}{3} \times 0.1 = 0.45\%.$$

as follows. First, we need to guess the equilibrium consumption c_1 . Based on c_1 , we can derive k_2 from the market clearing condition that is $c_1 + k_2 = f(k_1) + (1 - d)k_1$. From the capital market equilibrium condition, that is, $f'_t(k_t) = r_t + d$, we can derive r_t from k_t . c_2 can be derived from the Euler equation of *Eq.* (62). In this way, we can derive a sequence of $\{c_t\}_{t=1}^{\infty}$. If the sequence $\{c_t\}_{t=1}^{\infty}$ is not converging to the steady state, we repeat the search process with another guess of c_1 .¹³

We assume that in period 0, the economy is at a steady state with inflation volatility (σ). The government implements a stabilizing policy in period 1. In the simulation, we assume $(\sigma, \theta) = (3\%, 4)$, which corresponds to the inflation-volatility elasticity of an investment around 0.5. Figure 1 shows that from period 1, there will be an increase in the capital-to-output ratio (k_t/y_t) and it converges to 3 over time, which results in welfare improvement. Figure 1 also shows how consumption changes with the stabilizing policy compared to steady-state consumption without the policy. Consumption after the policy is initially lower than without a policy, but it becomes higher after year 14 due to the higher capital accumulation.

We measure the welfare gain from the stabilizing policy in terms of consumption of goods. The consumer is indifferent between equilibrium consumption without a stabilizing policy plus additional period-1 consumption subsidy (that is $m \times c_1^{\times}$ in *Eq.* (68)) and that with a stabilizing policy. Specifically, the measure of welfare gain (m) satisfies the following equation:

$$\begin{aligned} & u(c_1^{\times} (1 + m)) + \delta u(c_2^{\times}) + \delta^2 u(c_3^{\times}) + \dots \\ = & u(c_1^+) + \delta u(c_2^+) + \delta^2 u(c_3^+) + \dots \end{aligned} \tag{68}$$

where $(c_1^{\times}, c_2^{\times}, \dots)$ is equilibrium consumption without a stabilizing policy, and (c_1^+, c_2^+, \dots) is equilibrium with the policy. The numerical analysis shows that the welfare gain (m) is 41.91% if $(\sigma, \theta) = (3\%, 4)$.

We can derive the welfare gain (m) from a stabilizing policy in terms of inflation volatility (σ) and the risk-aversion parameter (θ) as shown in Table 2.

¹³The numerical analysis in this subsection was performed with MATLAB 9. All MATLAB codes can be downloaded from minwookkang.online/code/sunspots_rep.zip.

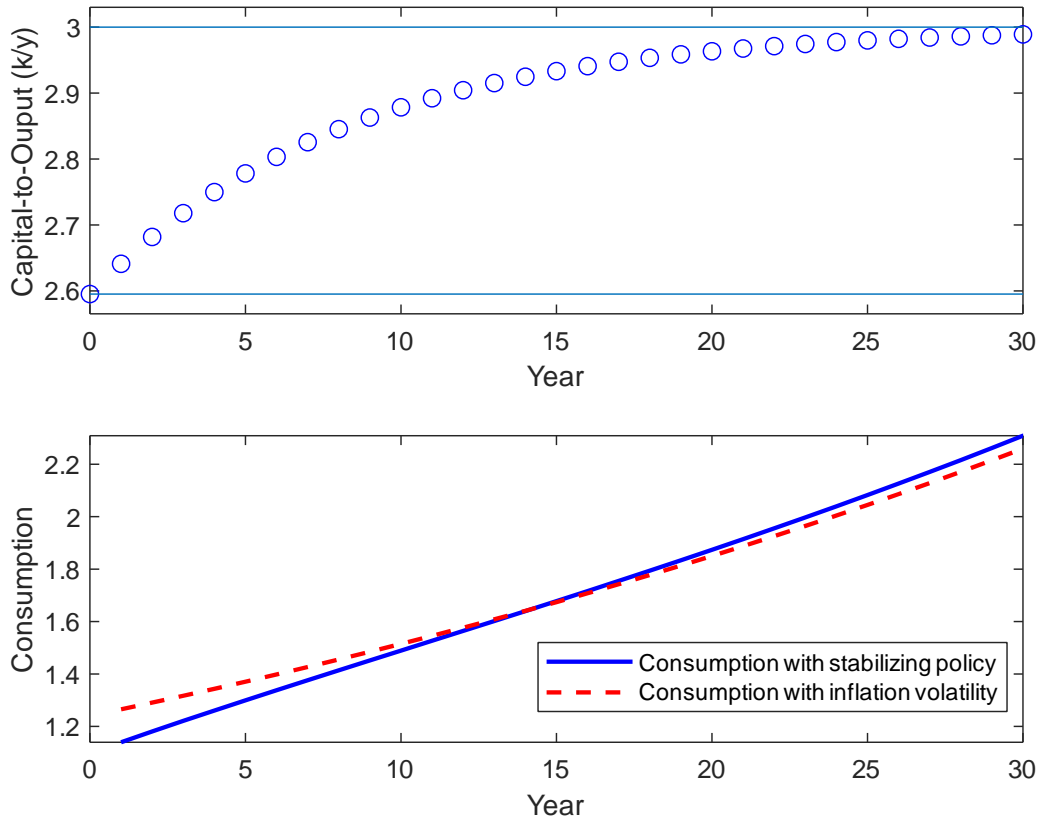


Figure 1: Steady-state convergence of capital-to-output ratio and consumption: The figure (up) plots the capital-to-output ratio changes over the years when $\alpha = 0.36, d = 0.08, g = 0.02, \beta = 0.981, \rho = 1$. In year 0, the capital-to-output ratio is 2.595 but it converges to 3 after the stabilizing policy is introduced in year 1. The figure (down) plots the consumption changes with (and without) the stabilizing policy, respectively.

		Inflation volatility (σ)		
		1%	3%	5%
Risk-aversion	2	1.45%	12.63%	35.63%
parameter	3	3.22%	28.63%	86.62%
(θ)	4	4.61%	41.91%	134.27%
	5	11.73%	122.49%	525.49%

Table 2. Welfare gain from the stabilizing policy as varying the risk-aversion parameter (θ) and relative standard deviation of inflation volatility (σ)

6. Conclusion

Much of the empirical evidence points to a negative relationship between inflation uncertainty and investment. However, there is no solid macroeconomic model that clearly explains the negative effect of inflation volatility on the economy. The literature also shows that through a self-fulfilling prophecy, higher inflation is strongly associated with a higher level of inflation uncertainty. When higher inflation is expected, people also naturally expect higher uncertainty about future price levels. This association between inflation and inflation uncertainty might imply that inflation uncertainty originates from market beliefs rather than from fundamental fluctuations. Based on the empirical evidence, this paper constructs a sunspot equilibrium model to provide a theoretical explanation of investigating inflation uncertainty and economic activity. The main result of this paper is that if a firm is risk averse, inflation uncertainty driven by sunspots has a negative impact on national investment and welfare. This postulates the importance of sunspot-stabilizing policies such as the introduction of indexed bonds due to their ability to enhance welfare.

Many previous sunspot-equilibrium models assume the existence of heterogeneous consumers, in which nominal volatility affects borrowers' and lenders' welfare in different ways. Specifically, Bhattacharya et al. (1998), Goenka and Prechac (2006), Kajii (2007) and Cozzi et al. (2017) show that some consumers can be better off with sunspots in a heterogeneous-agent model. In this case, the stabilizing policy can induce equilibrium allocations to be Pareto optimal but the policy may fail to improve the welfare of all consumers. However, in this paper, there is only one representative agent. Therefore, any stabilizing policy makes equilibrium allocation both Pareto optimal and Pareto superior.

The main result of this paper can be extended to an infinite-period model. The steady-state analysis shows that a sunspot-stabilizing policy can improve consumer welfare signifi-

cantly. When the stabilizing policy is initially applied, consumers would decrease consumption and increase investment. However, due to the higher capital accumulation, consumers will have a higher consumption level in the new steady state.

This paper does not consider intrinsic uncertainty. However, it would also be an interesting research extension to incorporate intrinsic uncertainty and analyze the separate effects of extrinsic and intrinsic uncertainty on asset allocation and welfare as shown in Manuelli and Peck (1992).¹⁴

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¹⁴Manuelli and Peck (1992) suggested that sunspot equilibria can be interpreted as a limiting case of markets overreacting to small shocks to fundamentals in an overlapping generation model.

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